

# A Class of Integrable Metrics

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## Abstract

In four dimensions, the most general metric admitting two Killing vectors and a rank-two Killing tensor can be parameterized by ten arbitrary functions of a single variable. We show that picking a special vierbien, reducing the system to eight functions, implies the existence of two geodesic and share-free, null congruences, generated by two principal null directions of the Weyl tensor. Thus, if the spacetime is an Einstein manifold, the Goldberg-Sachs theorem implies it is Petrov type D, and by explicit construction, is in the Carter class. Hence, our analysis provide an straightforward connection between the most general integrable structure and the Carter family of spacetimes.

## 1 Introduction and Discussion

The Kerr-(A)dS solution was discovered by Carter imposing the Einstein equations with a cosmological constant on a family of metrics identified by the requirement of the separability of the Schrödinger and Hamilton-Jacobi equations [1]. The mathematical structure behind the Hamilton-Jacobi separability on a spacetime with two Killing vectors is the existence of a Killing tensor (a modern review on the subject can be found in [2]). The most general  $D$ -dimensional metric with a rank-two Killing tensor and  $D - 2$  Killing vectors was found by Benenti and Francaviglia [3]. In the same paper, it is pointed out that the requirement of Schrödinger separability is redundant and that the new metrics contain the Carter metric as a special subcase.

The construction of the Carter form of the metric is heuristically explained in the lectures given in the “Les Houches Ecole d’Eté de Physique Théorique” [4]. Requiring the separability of the Klein-Gordon equation with a mass term, Carter ends up with an inverse metric of the form<sup>1</sup>:

$$\left(\frac{d}{ds}\right)^2 = \frac{1}{Z} \left\{ \Delta_x (\partial_x)^2 + \Delta_y (\partial_y)^2 + \frac{1}{\Delta_x} [Z_x (\partial_t) + C_x (\partial_\varphi)]^2 - \frac{1}{\Delta_y} [Z_y (\partial_t) + C_y (\partial_\varphi)]^2 \right\}, \quad (1)$$

with  $Z = C_y Z_x - C_x Z_y$ . The metric depends on four arbitrary functions of the coordinates  $(x, y)$ , namely  $\{Z_y, Z_x, \Delta_y, \Delta_x\}$ .  $C_x$  and  $C_y$  are constants. Indeed, the Carter ansatz (1) is a special case of the Benenti-Francaviglia metric:

$$g^{ab} \partial_a \partial_b = \frac{1}{S_1(x) + S_2(y)} \left[ \left( F_1^{ij}(x) - F_2^{ij}(y) \right) \partial_i \partial_j + \Delta_1(x) (\partial_x)^2 + \Delta_2(y) (\partial_y)^2 \right], \quad (2)$$

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<sup>1</sup>see eqs. (5.10)-(5.18) in [4]

where the indices  $a, b$  range over the coordinates  $\{\tau, \sigma, x, y\}$ , and the indices  $i, j$  run over  $\{\tau, \sigma\}$ .  $\{F_1^{ij} = F_1^{ji}, S_1, \Delta_i\}$  are five arbitrary functions depending on  $x$  and  $\{F_1^{ij} = F_1^{ji}, S_1, \Delta_i\}$  are five arbitrary functions depending on  $y$ . In this paper we show, by explicit calculation, that the system of equations of eight arbitrary functions, generated by replacing (2) in the Einstein equations with a cosmological constant can be fully integrated requiring only that:

$$F_1^{\tau\sigma} = \sqrt{F_1^{\tau\tau} F_1^{\sigma\sigma}} \quad \text{and} \quad F_2^{\tau\sigma} = \sqrt{F_2^{\tau\tau} F_2^{\sigma\sigma}}. \quad (3)$$

Furthermore, special attention is given to the existence of Killing-Yano (**KY**) tensors. In particular, we explicitly show how these tensors can be quite helpful in the integration of Einstein's equation. Moreover, KY tensors have proved to be a valuable tool in the study of black holes. Indeed, the analytic integration of the geodesic equation [1] as well as the Klein-Gordon and Dirac equations [5] in 4-dimensional Kerr spacetime is possible due to the existence of an integration constant constructed using the non-trivial Killing tensor of order two [6, 7]. Since this Killing tensor is the square of a KY tensor of order two [8], the integrability can be traced to the existence of a KY tensor. Likewise, KY tensors have proved to play a central role on the integrability of higher-dimensional black holes [9, 10]. Indeed, the class of Kerr-NUT-(A)dS spacetimes in arbitrary dimension admits a tower of KY tensors that enables the analytical integration of the geodesic equation [11, 12] along with the Klein-Gordon [13], and Dirac equations [14] in this background. KY tensors are also related to the separability of gravitational perturbations in these black holes [15, 16].

The interplay between supersymmetry and KY tensors have been discussed in the literature, see for instance [17]. Moreover, the Carter form of the metric has been used as the seed to find spinning solutions in gauged and ungauged supergravity [18]. The same form of the metric has been used to study the existence of supersymmetric solutions [19, 20]. The more general class (2), fits, in the string frame, the large family of rotating black holes which were recently found in the  $U(1)^4$  invariant sector of gauged  $\mathcal{N} = 8$  supergravity in four dimensions [21–25]. When multiplied by an arbitrary conformal factor, the metric (1) has been shown to be integrable in the presence of a real scalar field with an arbitrary scalar field self-interaction; the scalar field potential being integrated a posteriori and singled out by the form of the metric [26].

Therefore, it is worth to have at hand a systematic analysis of these ansätze in general cases. The results of section two, thus provide the conformal properties of the metric (2) under the condition (3). Namely, without imposing a field equation on the metric, the existence of geodesic and shear-free null-congruences is established by explicit construction. Later, using these conformal properties and the Goldberg-Sachs theorem we impose the Petrov type D condition on the metric. The remaining of the paper is dedicated to the integration of the Einstein equations with a cosmological constant, trying to be exhaustive in the analysis of subcases and existence of peculiar geometrical structures in every case.

The whole process can be done in the presence of a real scalar field with an arbitrary self-interaction along the lines of [26]. In this case, we found that the metric has to be conformally flat and that the scalar field and the spacetime are singular, we do not give the details of this result here. The paper leaves the door open to follow the study of the metric (2) without the condition (3). This is particularly interesting in the case when the cosmological constant is non-zero. No uniqueness theorem for the rotating black holes exists for asymptotically (anti)-de Sitter spacetimes [27].

## 2 The Conformal Properties of the metric

If a 4-dimensional spacetime possesses just two independent Killing vector fields then one can build three first integrals, two from the explicit symmetries and one by the metric, which is a Killing tensor.

Generally, these three constants of motion are not enough for an analytical integration of the geodesic equation. Nevertheless, if, besides the two Killing vectors and the metric, the spacetime has a non-trivial Killing tensor then one can build one extra first integral and the integration by quadratures of the geodesics is indeed possible. Moreover, this extra symmetry can also lead to the integrability of the Klein-Gordon and Dirac field equations in these backgrounds, as it happens to be the case with Kerr metric [1, 5]. Hence, we shall study the class of metrics with two commuting Killing vectors and one non-trivial Killing tensor of order two (2):

$$\mathbf{K} = \frac{-1}{S_1 + S_2} \left[ \left( F_1^{ij} S_2 + S_1 F_2^{ij} \right) \partial_i \partial_j + \Delta_1 S_2 (\partial_x)^2 - S_1 \Delta_2 (\partial_y)^2 \right]. \quad (4)$$

As we mentioned in the introduction, we will focus in the particular case where the following, degenerated vierbein exists:

$$F_1^{ij} \partial_i \partial_j = [f_1(x) \partial_\tau + h_1(x) \partial_\sigma]^2, \quad F_2^{ij} \partial_i \partial_j = [f_2(y) \partial_\tau + h_2(y) \partial_\sigma]^2. \quad (5)$$

Then, defining  $S(x, y) = S_1(x) + S_2(y)$  along with the vector fields

$$\mathbf{l} = \frac{1}{\sqrt{2S}} \left[ f_2 \partial_\tau + h_2 \partial_\sigma + \sqrt{\Delta_2} \partial_y \right], \quad (6)$$

$$\mathbf{n} = \frac{1}{\sqrt{2S}} \left[ f_2 \partial_\tau + h_2 \partial_\sigma - \sqrt{\Delta_2} \partial_y \right], \quad (7)$$

$$\mathbf{m}_1 = \frac{1}{\sqrt{2S}} \left[ f_1 \partial_\tau + h_1 \partial_\sigma + i \sqrt{\Delta_1} \partial_x \right], \quad (8)$$

$$\mathbf{m}_2 = \frac{1}{\sqrt{2S}} \left[ f_1 \partial_\tau + h_1 \partial_\sigma - i \sqrt{\Delta_1} \partial_x \right] \quad (9)$$

we have that the metric can be written as<sup>2</sup>:

$$\mathbf{g} = -\mathbf{l} \odot \mathbf{n} + \mathbf{m}_1 \odot \mathbf{m}_2. \quad (10)$$

So, we have that  $\{\mathbf{l}, \mathbf{n}, \mathbf{m}_1, \mathbf{m}_2\}$  is a null tetrad, namely the only non-vanishing inner products between the vectors of this basis are:

$$l^a n_a = -1 \quad \text{and} \quad m_1^a m_{2a} = 1.$$

Using this frame, the Killing tensor (4) can be conveniently written as:

$$\mathbf{K} = -S_1(x) \mathbf{l} \odot \mathbf{n} - S_2(y) \mathbf{m}_1 \odot \mathbf{m}_2. \quad (11)$$

The nice thing about this null tetrad is that, if we use the metric to transform the vector fields  $\{\mathbf{l}, \mathbf{n}, \mathbf{m}_1, \mathbf{m}_2\}$  into 1-forms, one can check that the following relations hold:

$$d\mathbf{l} \wedge \mathbf{l} \wedge \mathbf{m}_1 = 0 \quad \text{and} \quad d\mathbf{m}_1 \wedge \mathbf{l} \wedge \mathbf{m}_1 = 0, \quad (12)$$

$$d\mathbf{l} \wedge \mathbf{l} \wedge \mathbf{m}_2 = 0 \quad \text{and} \quad d\mathbf{m}_2 \wedge \mathbf{l} \wedge \mathbf{m}_2 = 0, \quad (13)$$

$$d\mathbf{n} \wedge \mathbf{n} \wedge \mathbf{m}_1 = 0 \quad \text{and} \quad d\mathbf{m}_1 \wedge \mathbf{n} \wedge \mathbf{m}_1 = 0, \quad (14)$$

$$d\mathbf{n} \wedge \mathbf{n} \wedge \mathbf{m}_2 = 0 \quad \text{and} \quad d\mathbf{m}_2 \wedge \mathbf{n} \wedge \mathbf{m}_2 = 0. \quad (15)$$

According to the Frobenius theorem, the first of these four relations guarantees that the surfaces orthogonal to  $\text{Span}\{\mathbf{l}, \mathbf{m}_1\}$  form a locally integrable foliation of the manifold. However, since the

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<sup>2</sup>In what follows, the symbol  $\odot$  stands for the symmetrized tensorial product of two vector fields. For instance,  $\mathbf{l} \odot \mathbf{n} = \mathbf{l} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{l}$ .

vectors  $\{\mathbf{l}, \mathbf{m}_1\}$  are both null and orthogonal to each other, it follows that these orthogonal surfaces are tangent to  $Span\{\mathbf{l}, \mathbf{m}_1\}$  itself. Since the tangent vectors to these surfaces are all null, and the maximum dimension of a null subspace in four dimensions is two, we say that  $Span\{\mathbf{l}, \mathbf{m}_1\}$  is a maximally isotropic integrable distribution. Analogously, the three remaining relations in (14) imply that  $Span\{\mathbf{l}, \mathbf{m}_2\}$ ,  $Span\{\mathbf{n}, \mathbf{m}_1\}$  and  $Span\{\mathbf{n}, \mathbf{m}_2\}$  are also maximally isotropic integrable distributions. In the Lorentzian case this is tantamount to saying that the vector fields  $\mathbf{l}$  and  $\mathbf{n}$  are both geodesic and shear-free [28].

Now, without loss of generality, one can write the functions  $f_1, h_1, f_2$  and  $h_2$  as follows:

$$f_1(x) = \frac{-P_1(x)}{\sqrt{A_1(x) \Delta_1(x)}} \quad , \quad h_1(x) = \frac{1}{\sqrt{A_1(x) \Delta_1(x)}} \quad , \quad (16)$$

$$f_2(y) = \frac{P_2(y)}{\sqrt{A_2(y) \Delta_2(y)}} \quad , \quad h_2(y) = \frac{1}{\sqrt{A_2(y) \Delta_2(y)}} \quad . \quad (17)$$

With these definitions the metric (2) has the following line element:

$$ds^2 = S \left[ \frac{-A_2 \Delta_2}{(P_1 + P_2)^2} (d\tau + P_1 d\sigma)^2 + \frac{A_1 \Delta_1}{(P_1 + P_2)^2} (d\tau - P_2 d\sigma)^2 + \frac{dx^2}{\Delta_1} + \frac{dy^2}{\Delta_2} \right] \quad . \quad (18)$$

In order to integrate Einstein's equation for the metric (18), it is useful to take advantage of the Goldberg-Sachs theorem. In its original version [29], such theorem states that a Ricci-flat 4-dimensional spacetime admits a geodesic and shear-free null congruence if, and only if, the Weyl tensor is algebraically special according to the Petrov classification with the repeated principal null direction being tangent to the shear-free congruence. Since the Weyl tensor, as well as the geodesic and shear-free property of a null congruence, are invariant under conformal transformations, it was soon realized that the Ricci-flat hypothesis could be weakened and replaced by a conformally invariant condition [30]. In particular, it was proved that the Goldberg-Sachs theorem also holds in the presence of a cosmological constant. Later, a version of the Goldberg-Sachs theorem valid in 4-dimensional manifolds of arbitrary signature was also proved [31]. Particularly, in non-Lorentzian signature the concept of geodesic and shear-free null congruence might be replaced by an integrable distribution of isotropic planes. Regarding the metric investigated here, (18), we have seen that the null vector fields  $\mathbf{l}$  and  $\mathbf{n}$  are geodesic and shear-free. Therefore, the Goldberg-Sachs theorem guarantees that whenever  $R_{ab} = \Lambda g_{ab}$  holds, with  $R_{ab}$  denoting the Ricci tensor,  $\mathbf{l}$  and  $\mathbf{n}$  will be repeated principal null directions of the Weyl tensor. In particular, this means that the Petrov type of the Weyl tensor is  $D$ . So, imposing the Weyl tensor of the metric (18) to be type  $D$  represents no constraint if Einstein's vacuum equation with a cosmological constant is assumed. Thus, our next step is to impose the type  $D$  condition to the metric (18).

Denoting the Weyl tensor by  $C_{abcd}$ , in the Lorentzian signature the components of the Weyl tensor can be assembled in the following five complex scalars [32]:

$$\Psi_0^+ \equiv C_{abcd} l^a m_1^b l^c m_1^d ; \quad \Psi_1^+ \equiv C_{abcd} l^a n^b l^c m_1^d ; \quad \Psi_2^+ \equiv C_{abcd} l^a m_1^b m_2^c n^d \quad (19)$$

$$\Psi_3^+ \equiv C_{abcd} l^a n^b m_2^c n^d ; \quad \Psi_4^+ \equiv C_{abcd} n^a m_2^b n^c m_2^d \quad . \quad (20)$$

Computing these scalars for the metric (18) we find that  $\Psi_0 = 0 = \Psi_4$ , which means that  $\mathbf{l}$  and  $\mathbf{n}$  are principal null directions of the Weyl tensor. In this case, the type  $D$  constraint amounts to imposing both  $\Psi_1$  and  $\Psi_3$  to vanish. However, one can check that for this line element the relation  $\Psi_1 = \Psi_3$  holds, so that we just need to impose  $\Psi_1$  to vanish. Solving this constraint, we find that the Petrov type of metric (18) is  $D$  if, and only if,  $A_1$  and  $A_2$  can be put in the following form:

$$A_1(x) = \frac{(P'_1)^2}{4 (b_1 P_1 + \eta_1) (b_2 P_1 + \eta_2)} \quad , \quad A_2(y) = \frac{-(P'_2)^2}{4 (b_1 P_2 - \eta_1) (b_2 P_2 - \eta_2)} \quad . \quad (21)$$

Where  $P'_1$  and  $P'_2$  stand for the first derivatives of  $P_1(x)$  and  $P_2(y)$  respectively, whereas  $b_1, b_2, \eta_1$  and  $\eta_2$  are arbitrary constants. Note that if  $b_2 \neq 0$  then one can always absorb a multiplicative factor in the other constants and make  $b_2 = 1$ . In spite of such freedom, for reasons of aesthetic symmetry, we shall not take advantage of this possibility. It is worth stressing that the above expressions are not valid if either  $P_1$  or  $P_2$  are constant functions, since in this case  $A_1$  or  $A_2$  would vanish according to (21), which would imply the determinant of the metric to vanish. Indeed, one can check that if  $P'_1 = 0$  and  $P'_2 \neq 0$  then, in order for the metric to be type  $D$ , the function  $A_1(x)$  can be arbitrary while  $A_2(y)$  might be given by

$$A_2(y) = \frac{c (P'_2)^2}{(P_1 + P_2)^2} \quad (P_1 = \text{constant}), \quad (22)$$

with  $c$  being a non-zero constant. Analogously, if  $P'_2 = 0$  and  $P'_1 \neq 0$  then, in order for the Petrov classification to be type  $D$ , the function  $A_2(y)$  can be arbitrary while  $A_1(x)$  might be given by

$$A_1(x) = \frac{c (P'_1)^2}{(P_1 + P_2)^2} \quad (P_2 = \text{constant}). \quad (23)$$

Finally, if  $P_1$  and  $P_2$  are both constant then the metric (21) is automatically type  $D$ . In forthcoming sections Einstein's equation for the metric (21) will be fully integrated and the type  $D$  condition will be helpful for the achievement of this goal. We shall separate our analysis in three cases depending on whether the functions  $P_1(x)$  and  $P_2(y)$  are constant or not.

As an aside, it is worth noting that along these calculations to impose the type  $D$  condition it was implicitly assumed that the signature is Lorentzian. In the non-Lorentzian case the self-dual and the anti-self-dual parts of the Weyl tensor are not related to each other by complex conjugation, so that besides the five Weyl scalars defined in (20) one must also consider following other five scalars [32]:

$$\Psi_0^- \equiv C_{abcd} l^a m_2^b l^c m_2^d; \quad \Psi_1^- \equiv C_{abcd} l^a n^b l^c m_2^d; \quad \Psi_2^- \equiv C_{abcd} l^a m_2^b m_1^c n^d \quad (24)$$

$$\Psi_3^- \equiv C_{abcd} l^a n^b m_1^c n^d; \quad \Psi_4^- \equiv C_{abcd} n^a m_1^b n^c m_1^d. \quad (25)$$

In spite of this further complication in the non-Lorentzian case, one can check that the above restrictions for the functions  $A_1(x)$  and  $A_2(y)$  also imply that the anti-self-dual part of the Weyl tensor is type  $D$ , namely the Weyl scalars  $\Psi_0^-, \Psi_1^-, \Psi_3^-$  and  $\Psi_4^-$  vanish simultaneously. Thus, for an arbitrary signature, the conditions (21), (22) and (23) imply that the algebraic type of the Weyl tensor is  $(D, D)$  according to the generalized Petrov classification [32].

### 3 Integrating Einstein's Equation for the General Case

In the present section let us deal with the general case in which  $P_1(x)$  and  $P_2(y)$  are both non-constant functions. In this case, one can define new coordinates  $\hat{x} = \sqrt{P_1(x)}$  and  $\hat{y} = \sqrt{P_2(y)}$  and then judiciously redefine  $A_1, \Delta_1, A_2$  and  $\Delta_2$  in such a way that, omitting the hats, the line element (18) becomes:

$$ds^2 = S \left[ \frac{-A_2 \Delta_2}{(x^2 + y^2)^2} (d\tau + x^2 d\sigma)^2 + \frac{A_1 \Delta_1}{(x^2 + y^2)^2} (d\tau - y^2 d\sigma)^2 + \frac{dx^2}{\Delta_1} + \frac{dy^2}{\Delta_2} \right]. \quad (26)$$

The goal of this section is to solve Einstein's vacuum equation for the metric (26), namely we shall integrate the equation

$$R_{ab} = \Lambda g_{ab}. \quad (27)$$

As explained in the preceding section, if (27) holds then the algebraic type of the Weyl tensor for the metric considered here is  $D$ , so that the functions  $A_1$  and  $A_2$  are given by (21). In particular, since we have chosen a gauge in which  $P_1(x)$  and  $P_2(y)$  are  $x^2$  and  $y^2$  respectively, it follows that

$$A_1(x) = \frac{x^2}{(b_1 x^2 + \eta_1)(b_2 x^2 + \eta_2)} \quad \text{and} \quad A_2(y) = \frac{-y^2}{(b_1 y^2 - \eta_1)(b_2 y^2 - \eta_2)}. \quad (28)$$

It is worth noting that if  $S = x^2 + y^2$  and  $A_1 = 1 = A_2$  ( $b_1 = 0 = \eta_2$  and  $\eta_1 = 1 = b_2$ ), the above line element reduces to the canonical form of Carter's metric [1]. Particularly, assuming  $S = x^2 + y^2$  along with  $A_1 = 1 = A_2$  and then solving Einstein's vacuum equation with a cosmological constant we are lead to Kerr-NUT-(A)dS metric [33]. In the present article, we shall go one step further and integrate Einstein's vacuum equation with a cosmological constant for the full metric (26), with  $S = S_1(x) + S_2(y)$ ,  $A_1(x)$ ,  $A_2(y)$ ,  $\Delta_1(x)$  and  $\Delta_2(y)$  being, in principle, arbitrary functions.

Einstein's vacuum equation,  $R^a_b = \Lambda \delta^a_b$ , implies that  $R^x_y = 0$  which, in turn, is equivalent to the following differential equation:

$$4xy(S_1 + S_2)^2 = (x^2 + y^2)^2 \frac{dS_1}{dx} \frac{dS_2}{dy}. \quad (29)$$

Working out the general solution of (29) yields

$$S_1(x) = \frac{b_3 x^2 + \eta_3}{b_4 x^2 + \eta_4} \quad , \quad S_2(y) = -\frac{b_3 y^2 - \eta_3}{b_4 y^2 - \eta_4}. \quad (30)$$

Where  $b_3$ ,  $b_4$ ,  $\eta_3$  and  $\eta_4$  are arbitrary constants. Now, inserting (28) and (30) into the equation  $R_{ab}l^a l^b = 0$  one can see that one of the following relations must hold:

$$b_4 \eta_1 - b_1 \eta_4 = 0 \quad \text{or} \quad b_4 \eta_2 - b_2 \eta_4 = 0. \quad (31)$$

Assuming that  $b_4 \neq 0$ , we can set  $b_4 = b_1$  in (30) by redefinition of the other integration constants. Thus, it follows that, up to a permutation of the integration constants that  $b_4 \neq 0$  and  $R_{ab}l^a l^b = 0$  implies  $\eta_4 = \eta_1$ . Therefore, the conformal factor  $S_1(x) + S_2(y)$  is given in terms of the functions

$$S_1(x) = \frac{b_3 x^2 + \eta_3}{b_1 x^2 + \eta_1} \quad , \quad S_2(y) = -\frac{b_3 y^2 - \eta_3}{b_1 y^2 - \eta_1}. \quad (32)$$

Assuming (28) and (32) to hold we have that the following eight components of Einstein's vacuum equation are immediately satisfied:

$$R_{ab}l^a l^b = R_{ab}n^a n^b = R_{ab}m_1^a m_1^b = R_{ab}m_2^a m_2^b = 0, \quad (33)$$

$$R_{ab}l^a m_1^b = R_{ab}l^a m_2^b = R_{ab}n^a m_1^b = R_{ab}n^a m_2^b = 0. \quad (34)$$

Hence, it just remains to integrate the equations  $R_{ab}m_1^a m_2^b = \Lambda$  and  $R_{ab}l^a n^b = -\Lambda$ , which yield a coupled system of linear differential equations for  $\Delta_1(x)$  and  $\Delta_2(y)$  whose general solution is

$$\begin{aligned} \Delta_1(x) &= \frac{I_1 J_1}{x^2} \left[ d_1 I_1^{3/2} J_1^{1/2} + d_2 I_1^2 + d_3 I_1 J_1 + \frac{\Lambda}{3b_1^2} \frac{b_1 \eta_3 - b_3 \eta_1}{b_2 \eta_1 - b_1 \eta_2} \right], \\ \Delta_2(y) &= \frac{I_2 J_2}{y^2} \left[ d_4 I_2^{3/2} J_2^{1/2} - d_2 I_2^2 - d_3 I_2 J_2 - \frac{\Lambda}{3b_1^2} \frac{b_1 \eta_3 - b_3 \eta_1}{b_2 \eta_1 - b_1 \eta_2} \right]. \end{aligned} \quad (35)$$

Where the  $d$ 's are arbitrary constants and  $I_1$ ,  $I_2$ ,  $J_1$  and  $J_2$  represent the following functions:

$$I_1(x) = b_1 x^2 + \eta_1 \quad , \quad J_1(x) = b_2 x^2 + \eta_2 \quad , \quad (36)$$

$$I_2(y) = b_1 y^2 - \eta_1 \quad , \quad J_2(y) = b_2 y^2 - \eta_2 \quad . \quad (37)$$

Thus, we have completely integrated Einstein's vacuum equations with a cosmological constant for the metric (26), the general solution being given by (28), (32), and (35). Actually, this is, locally, the Kerr-NUT-de Sitter metric, as can be seen by the change of coordinates  $(x, y) \rightarrow (p, q)$

$$x^2 = b_1^{-1} \left( p^2 - \frac{b_2}{b_1 \eta_2 - b_2 \eta_1} \right)^{-1} - b_1^{-1} \eta_1 \quad , \quad y^2 = b_1^{-1} \left( q^2 + \frac{b_2}{b_1 \eta_2 - b_2 \eta_1} \right)^{-1} + b_1^{-1} \eta_1 \quad , \quad (38)$$

and a relabeling of the integration constants.

## 4 Killing-Yano Tensors

A totally skew-symmetric tensor of rank  $p$ ,  $Y_{a_1 a_2 \dots a_p} = Y_{[a_1 a_2 \dots a_p]}$ , is called a Killing-Yano (KY) tensor of order  $p$  whenever it obeys the following generalization of the Killing vector equation:

$$\nabla_a Y_{b_1 b_2 \dots b_p} + \nabla_{b_1} Y_{a b_2 \dots b_p} = 0 \quad . \quad (39)$$

By means of a KY tensor one can build objects that are conserved along the geodesic motion. Indeed, if  $Y_{a_1 a_2 \dots a_p}$  is a Killing-Yano tensor and  $T^a$  is an affinely parameterized geodesic vector field,  $T^a \nabla_a T_b = 0$ , then the tensor  $T^a Y_{a b_2 \dots b_p}$  is constant along each geodesic curve tangent to  $\mathbf{T}$ . As a consequence, the scalar  $Y_a{}^{c_2 \dots c_p} Y_{c_2 \dots c_p b} T^a T^b$  is also conserved along the geodesics tangent to  $\mathbf{T}$ . This, in turn, means that the symmetric tensor

$$Q_{ab} = Y_a{}^{c_2 \dots c_p} Y_{c_2 \dots c_p b} \quad (40)$$

is a Killing tensor of order two. Thus, to each KY tensor it is associated a Killing tensor of order two, although the converse generally is not true, as we shall see. Because of this, one can say that KY tensors are, in a sense, more fundamental than Killing tensors. Physically, this is corroborated by the fact that classical symmetries associated to KY tensors are preserved at the quantum level, whereas those associated to Killing tensors generally are not [34]. In this section we shall investigate whether the Killing tensor of our metric (26) is the square of a Killing-Yano tensor. For a detailed discussion of KY tensors in 4-dimensional spacetimes the reader is referred to [35, 36].

Since  $\partial_\tau$  and  $\partial_\sigma$  are Killing vector fields and the metric is covariantly constant, it follows that the most general Killing tensor of order two in a manifold with line element (18) is given by

$$\mathbf{Q} = \alpha \mathbf{K} + \beta \mathbf{g} + \gamma^{ij} \partial_i \odot \partial_j \quad , \quad (41)$$

where  $\mathbf{K}$  is given by (4),  $\mathbf{g}$  is the metric tensor and the coefficients  $\alpha$ ,  $\beta$  and  $\gamma^{ij}$  are arbitrary constants. Now, for simplicity, let us neglect the terms of  $\mathbf{Q}$  coming from the symmetrized products of Killing vectors, *i.e.*, set  $\gamma^{ij} = 0$ . Then, using (10) along with (11) lead us to:

$$\mathbf{Q} = \alpha \mathbf{K} + \beta \mathbf{g} = -(\alpha S_1 + \beta) \mathbf{l} \odot \mathbf{n} - (\alpha S_2 - \beta) \mathbf{m}_1 \odot \mathbf{m}_2 \quad . \quad (42)$$

The goal of the present section is to look for the existence of a Killing-Yano tensor whose square have the form of the Killing tensor (42). In this section we shall work with the general metric (26) without restricting the functions  $A_1$ ,  $A_2$ ,  $\Delta_1$ ,  $\Delta_2$ ,  $S_1$  and  $S_2$ . In particular, Einstein's equation will not be assumed to hold.



Since a KY tensor of order one is just a Killing vector and, in four dimensions, a KY tensor of order four is a constant multiple of the volume-form, it follows that the only non-trivial Killing-Yano tensors are the ones of order two and three. Let us first consider the possibility of  $\mathbf{Q}$  being the square of a KY tensor of order three. In this case,  $\mathbf{Q}$  would have the following form [37]:

$$Q_{ab} = \xi_a \xi_b - (\xi^c \xi_c) g_{ab}, \quad (43)$$

where  $\xi_a$  is a conformal Killing vector. However, expanding the vector  $\xi$  in the null tetrad basis and then inserting into (43) one can easily see that (43) is equal to (42) only in the trivial case in which  $\xi_a = 0$  and  $\alpha = 0 = \beta$ . Thus, a non-zero  $\mathbf{Q}$  cannot be the square of a KY tensor of order three. It remains to check whether Killing tensor  $\mathbf{Q}$  in (42) is the square of a KY tensor of order two.

If a bivector  $Y_{ab}$  is such that its square has the algebraic form of  $\mathbf{Q}$  in Eq. (42) then it might have the following form:

$$\mathbf{Y} = -\Phi_1 \mathbf{l} \wedge \mathbf{n} + i \Phi_2 \mathbf{m}_1 \wedge \mathbf{m}_2 \quad (44)$$

where

$$(\Phi_1)^2 = \alpha S_1 + \beta \quad \text{and} \quad (\Phi_2)^2 = \alpha S_2 - \beta. \quad (45)$$

Since the integrability condition for the existence of a Killing-Yano tensor of order two implies that the Petrov type of the Weyl tensor is  $D$ ,  $N$  or  $O$ , we can attain ourselves to these cases. Nevertheless, since the Weyl scalars of the metric (18) are such that  $\Psi_0 = \Psi_4 = 0$  it follows that the type  $N$  is forbidden. Then, using the fact that the type  $O$  can be seen as a special case of the type  $D$ , we conclude that a necessary condition for  $\mathbf{Y}$  to be a KY tensor is that the Weyl tensor should be at least type  $D$ . Therefore, without loss of generality, we can assume (28) to hold whenever the space with metric (26) admits a KY tensor. Then, by means of integrating the Killing-Yano equation for the bivector (44) in a space with the general metric (26) along with (28), we see that: besides  $\Phi_1$  and  $\Phi_2$ , we have that the functions  $S_1$  and  $S_2$  appearing in the metric are also constrained, which can be grasped from the relation (45). The final result is that  $\mathbf{Y}$  is a KY tensor if, and only if, the functions  $\Phi_1$ ,  $\Phi_2$ ,  $S_1$  and  $S_2$  are given by:

$$\Phi_1(x) = c \sqrt{\frac{b_2 x^2 + \eta_2}{b_1 x^2 + \eta_1}} \quad , \quad \Phi_2(y) = c \sqrt{\frac{-b_2 y^2 + \eta_2}{b_1 y^2 - \eta_1}} \quad (46)$$

$$S_1(x) = \frac{b_3 x^2 + \eta_3}{b_1 x^2 + \eta_1} \quad , \quad S_2(y) = -\frac{b_3 y^2 - \eta_3}{b_1 y^2 - \eta_1}. \quad (47)$$

Where  $c$ ,  $b_3$  and  $\eta_3$  are arbitrary constants,<sup>3</sup> whereas the constants  $b_1$ ,  $b_2$ ,  $\eta_1$  and  $\eta_2$  are the ones appearing in (28). It is interesting noting that the functions  $S_1$  and  $S_2$  compatible with the existence of a KY tensor of order two are exactly equal to the ones found while solving Einstein's vacuum equation, see (32). In particular, this means that the requirement of the existence of a Killing-Yano tensor in a space with line element (26) implies that the eight components (34) of Einstein's vacuum equation are satisfied. This hints that often the geometrical requirement of the existence of a KY tensor might be quite helpful in integrating Einstein's vacuum equation. Particularly, (47) implies that all the vacuum solutions found in Sec. 3 are endowed with a Killing-Yano tensor. Indeed, it is well-known that all type  $D$  Ricci-flat spacetimes possessing a non-trivial Killing tensor also have a KY tensor [8]. The results of this section illuminates the possibility that the latter fact can be extended from Ricci-flat to Einstein spacetimes, namely to the case of non-zero cosmological constant.

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<sup>3</sup>Besides the solution displayed in Eqs. (46) and (47), one also have a solution if the replacements  $b_1 \leftrightarrow b_2$  and  $\eta_1 \leftrightarrow \eta_2$  are performed in (46) and (47). However, since this other solution can be obtained from the previous one just by a redefinition of constants that are not fixed yet, we shall consider that they represent the same solution.



The square of the Killing-Yano tensor  $\mathbf{Y}$ ,  $Q_{ab} = Y_a{}^c Y_{cb}$ , is given by

$$\mathbf{Q} = -(\Phi_1)^2 \mathbf{l} \odot \mathbf{n} - (\Phi_2)^2 \mathbf{m}_1 \odot \mathbf{m}_2. \quad (48)$$

Comparing (11) and (48) we conclude that in order to have  $\mathbf{K} = \mathbf{Q}$  the relations  $(\Phi_1)^2 = S_1$  and  $(\Phi_2)^2 = S_2$  must both hold. However, in general we cannot manage to choose the constant  $c$  appearing in (46) to be such that the latter conditions are satisfied. Which lead us to the conclusion that generally there is no KY tensor whose square is the Killing tensor  $\mathbf{K}$ . Instead, the square of the KY tensor  $\mathbf{Y}$  is a linear combination of  $\mathbf{K}$  and  $\mathbf{g}$ , as anticipated in Eq. (42).

Note that the functions  $\Delta_1$  and  $\Delta_2$  are not constrained by the Killing-Yano equation. So, there are non-vacuum type  $D$  spacetimes that admit the existence of a KY tensor, which is already clear in Carter's metric [1, 33]. What maybe is not so clear in the literature and is clarified by our results is that there are type  $D$  spacetimes admitting a non-trivial Killing tensor that do not admit KY tensors. Indeed, if  $A_1$  and  $A_2$  are given by (28) and the functions  $S_1$  and  $S_2$  are not of the form displayed in (47) then the spacetime with metric (26) is type  $D$ , posses a non-trivial Killing tensor but does not admit a KY tensor.

## 5 Integrating Einstein's Equation when $P'_1 \neq 0$ and $P'_2 = 0$

In the previous sections we considered the metric (18) in the general case when both functions  $P_1$  and  $P_2$  are non-constant. Now, we shall investigate the cases in which at least one of these functions are constant. Particularly, the aim of the present section is to study the case of  $P'_1 \neq 0$  and  $P'_2 = 0$ . More precisely, we shall fully integrate Einstein's vacuum equation with a cosmological constant and look for the existence of Killing-Yano tensors in these solutions. Note that it is needless to consider the analogous case  $P'_1 = 0$  and  $P'_2 \neq 0$ , inasmuch as such a case can be easily obtained from the case  $P'_1 \neq 0$  and  $P'_2 = 0$  by interchanging the coordinates  $x$  and  $y$ .

Since in this section it will be assumed that  $P_1(x)$  is non-constant, it follows that we can redefine the coordinate  $x$ , along with the functions  $A_1$  and  $\Delta_1$ , in such a way that  $P_1(x) = \frac{1}{x} - p_2$ , with  $p_2$  denoting the constant value of the function  $P_2(y)$ . Then, using this gauge choice, the line element (18) becomes

$$ds^2 = S \left[ -x^2 A_2 \Delta_2 (d\tau + (x^{-1} - p_2) d\sigma)^2 + A_1 \Delta_1 x^2 (d\tau - p_2 d\sigma)^2 + \frac{dx^2}{\Delta_1} + \frac{dy^2}{\Delta_2} \right] \quad (49)$$

where  $S = S_1(x) + S_2(y)$ . Furthermore, using the coordinate  $\phi = \frac{1}{\ell}(\tau - p_2\sigma)$  instead of  $\tau$ , the line element assumes the following form:

$$ds^2 = S \left[ -A_2 \Delta_2 (d\sigma + x \ell d\phi)^2 + x^2 A_1 \Delta_1 \ell^2 d\phi^2 + \frac{dx^2}{\Delta_1} + \frac{dy^2}{\Delta_2} \right], \quad (50)$$

where  $\ell$  is a non-zero constant introduced for future convenience. Now, let us integrate Einstein's vacuum equation for the above line element.

As explained in Sec. 2, a necessary condition for the above metric to be a solution of Einstein's vacuum equation is that the Weyl tensor might have Petrov type  $D$ . According to (23), the type  $D$  condition holds if, and only if,  $A_1$  takes the following form

$$A_1(x) = \frac{1}{\ell^2 x^2}. \quad (51)$$

Inserting (51) into (50) and then computing the Ricci tensor we find that:

$$R^x{}_y = \frac{3 \Delta_1 S'_1 S'_2}{2 (S_1 + S_2)^3}, \quad (52)$$

where, as usual, the primes denote that the function is being differentiated with respect to its variable. Then, imposing  $R^a_b = \Lambda \delta^a_b$  we conclude that the right hand side of (52) must vanish. Thus, either  $S_1(x)$  or  $S_2(y)$  might be constant. In principle, one could also have that both  $S_1(x)$  or  $S_2(y)$  are constant. However, assuming  $S_2(y)$  to be constant we find that  $R_{ab}l^a l^b$  does not vanish as it should. Therefore, we conclude that  $S_2(y)$  should be a non-constant function, while  $S_1(x)$  is a constant that we shall denote by  $s_1$ . Thus, the conformal factor  $S(x, y)$  should be just a function of  $y$ :

$$S(x, y) = s_1 + S_2(y) = S(y). \quad (53)$$

Now, without loss of generality, let us choose the coordinate  $y$  in such a way that

$$S(y) = y^2 + n_1^2 \quad (54)$$

with  $n_1$  being a constant. Since the value of  $n_1$  can be shifted by means of redefining the coordinate  $y$ , in what follows it will be assumed that  $n_1 \neq 0$ . Then, assuming (54) and imposing  $R_{ab}l^a l^b$  to vanish it follows that  $A_2(y)$  must be given by:

$$A_2(y) = \frac{4y^2}{(4n_1^2 n_2^2 - \ell^2 + 4n_2^2 y^2)(n_1^2 + y^2)^2}, \quad (55)$$

with  $n_2$  being an integration constant. Postponing the special case  $n_2 = 0$  to the forthcoming section, let us assume  $n_2 \neq 0$ . In the latter case we can choose the non-zero parameter  $\ell$  to be equal to  $2n_1 n_2$ , in which case we have

$$A_2(y) = \frac{1}{n_2^2 (n_1^2 + y^2)^2}. \quad (56)$$

Once assumed the latter expression for  $A_2$ , the eight components of Einstein's vacuum equation displayed in (34) are immediately satisfied. Finally, imposing the equation  $R_{ab}m_1^a m_2^b = \Lambda$  we find that the functions  $\Delta_1$  and  $\Delta_2$  should have the following general form:

$$\Delta_1(x) = -a_2 x^2 + a_1 x + a_0, \quad (57)$$

$$\Delta_2(y) = \left( n_1^4 - 2n_1^2 y^2 - \frac{1}{3} y^4 \right) \Lambda + a_2 (y^2 - n_1^2) + b y, \quad (58)$$

with  $a_0$ ,  $a_1$ ,  $a_2$  and  $b$  being integration constants. In particular,  $b$  is related to the ADM mass of the solution. In conclusions, the general solution of Einstein's vacuum equation with a cosmological constant for the metric (18) with  $P_2$  constant and  $P_1$  non-constant is given by the equations (50), (51), (54), (56), (57) and (58). It turns out that such solution posses four Killing vector fields. Indeed, defining  $\omega = \frac{1}{2}\sqrt{a_1^2 + 4a_0 a_2}$ , it can be verified that that the following vector fields generate isometries:

$$\chi_1 = -\sin(\omega\phi) \frac{n_1 n_2 (2a_0 + a_1 x)}{\omega \sqrt{\Delta_1}} \partial_\sigma + \sin(\omega\phi) \frac{2a_2 x - a_1}{2\omega \sqrt{\Delta_1}} \partial_\phi + \cos(\omega\phi) \sqrt{\Delta_1} \partial_x \quad (59)$$

$$\chi_2 = \cos(\omega\phi) \frac{n_1 n_2 (2a_0 + a_1 x)}{\omega \sqrt{\Delta_1}} \partial_\sigma - \cos(\omega\phi) \frac{2a_2 x - a_1}{2\omega \sqrt{\Delta_1}} \partial_\phi + \sin(\omega\phi) \sqrt{\Delta_1} \partial_x, \quad (60)$$

in addition to the obvious Killing vector fields  $\chi_3 = \partial_\phi$  and  $\chi_4 = \partial_\sigma$ .

Since  $S = S_1 + S_2 = s_1 + S_2$ , it follows that the we can absorb the constant  $s_1$  into the function  $S_2$  so that instead of using the functions  $S_1$  and  $S_2$  one could equivalently use  $\tilde{S}_1(x) = 0$  and  $\tilde{S}_2(y) = s_1 + S_2 = S$ . Thus, besides the Killing tensor (11), we expect that the tensor

$$\mathbf{K}_2 = -\tilde{S}_1(x) \mathbf{l} \odot \mathbf{n} - \tilde{S}_2(y) \mathbf{m}_1 \odot \mathbf{m}_2 = -S(y) \mathbf{m}_1 \odot \mathbf{m}_2 \quad (61)$$

should also be a Killing tensor. Indeed, this can be readily verified. However, it turns out that this new Killing tensor does not lead to new conserved charges, which can be grasped from the fact that  $\mathbf{K}_2$  is just a linear combination of  $\mathbf{K}$  and the metric,  $\mathbf{K}_2 = \mathbf{K} - s_1 \mathbf{g}$ . Moreover, the latter Killing tensor is reducible, in the sense that it can be written in terms of symmetrized products of Killing vectors. Indeed, one can check that

$$\mathbf{K}_2 = \frac{2a_0 n_1^2 n_2^2}{\omega^2} \chi_4 \odot \chi_4 + \frac{a_1 n_1 n_2}{\omega^2} \chi_4 \odot \chi_3 - \frac{a_2}{2\omega^2} \chi_3 \odot \chi_3 - \frac{1}{2} \chi_1 \odot \chi_1 - \frac{1}{2} \chi_2 \odot \chi_2. \quad (62)$$

The solution found in this section also posses a Killing-Yano tensor given by

$$\mathbf{Y} = n_1 \mathbf{l} \wedge \mathbf{n} + i y \mathbf{m}_1 \wedge \mathbf{m}_2, \quad (63)$$

whose square is  $\mathbf{K}_2 + n_1^2 \mathbf{g}$ .

Regarding the interpretation of the latter metric, the existence of four Killing vectors hints the existence of spherical symmetry and that such solution might be a generalization of the Schwarzschild metric. Indeed, if  $a_2 \neq 0$ , it follows that the killing vector fields

$$\tilde{\chi}_1 = \frac{1}{a_2} \chi_1, \quad \tilde{\chi}_2 = \frac{1}{\sqrt{a_2}} \chi_2, \quad \tilde{\chi}_3 = -\frac{1}{\Lambda \sqrt{a_2}} \left( \chi_3 - \frac{a_1 n_1 n_2}{a_2} \chi_4 \right), \quad (64)$$

generate the  $SO(3)$  Lie algebra

$$[\tilde{\chi}_i, \tilde{\chi}_j] = \varepsilon_{ij}^{\quad k} \tilde{\chi}_k, \quad (65)$$

with  $\varepsilon_{ij}^{\quad k}$  denoting the usual Levi-Civita symbol. Therefore, in the case  $a_2 \neq 0$ , the isometry group is  $SO(3) \times \mathbb{R}$ , with  $\chi_4$  spanning the center of the algebra. This gives a clue that the Taub-NUT solution with a cosmological constant might be contained in the class of metrics that we have just found. Indeed, assuming  $a_1 = 0$  and defining new coordinates  $\{t, y, \theta, \varphi\}$  by  $\sigma = n_2 t$ ,  $x = \sqrt{\frac{a_0}{a_2}} \cos \theta$  and  $\varphi = -\sqrt{a_0 a_2} \phi$  we have that the metric can be written as

$$ds^2 = -\frac{\Delta_2}{y^2 + n_1^2} \left( dt - \frac{2n_1}{a_2} \cos \theta d\varphi \right)^2 + \frac{y^2 + n_1^2}{\Delta_2} dy^2 + \frac{y^2 + n_1^2}{a_2} (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (66)$$

with  $\Delta_2(y)$  given by (58). Making  $a_2 = 1$  and  $\Lambda = 0$  we get the Taub-NUT solution in the form presented in [41], with  $n_1$  being the NUT parameter and  $-b/2$  being the mass. On the other hand, in the special case in which  $a_2 = 0$ , the isometry Lie algebra is not the direct sum of an abelian algebra and a semi-simple Lie algebra. Indeed, in such a case we have that

$$[\chi_1, \chi_3] = \frac{a_1}{2} \chi_2, \quad [\chi_2, \chi_3] = -\frac{a_1}{2} \chi_1, \quad [\chi_1, \chi_2] = 2n_1 n_2 \chi_4, \quad (67)$$

with all other commutators being zero.

### 5.1 The special case $n_2 = 0$

Now, let us consider the special case in which the integration constant  $n_2$  vanishes. In such a case, Eq. (55) gives

$$A_2(y) = \frac{-4y^2}{\ell^2 (n_1^2 + y^2)^2}. \quad (68)$$

Assuming the latter expression for  $A_2$ , the eight components of Einstein's vacuum equation displayed in (34) are immediately satisfied. Finally, imposing the equation  $R_{ab} m_1^a m_2^b = \Lambda$  we find that the functions  $\Delta_1$  and  $\Delta_2$  might have the following general form:

$$\Delta_1(x) = -a_2 x^2 + a_1 x + a_0 \quad (69)$$

$$\Delta_2(y) = \frac{b}{y^2} - \frac{\Lambda}{2} \left( n_1^4 + n_1^2 y^2 + \frac{1}{3} y^4 \right) + \frac{a_2}{4} (2 n_1^2 + y^2) , \quad (70)$$

One can check that the functions  $A_2$  and  $\Delta_2$  can be conveniently written in terms of the function  $S(y)$  as follows:

$$A_2(y) = - \left( \frac{S'}{\ell S} \right)^2 , \quad \Delta_2(y) = \frac{1}{(S')^2} \left[ \tilde{b} + a_2 S^2 - \frac{2}{3} \Lambda S^3 \right] . \quad (71)$$

Where  $\tilde{b} \equiv (4b - a_2 n_1^4 + \frac{2}{3} \Lambda n_1^6)$  is a constant that replaces the arbitrary constant  $b$ . It is worth pointing out that the functions  $A_2$  and  $\Delta_2$  as written in (71) provide a solution for Einstein's vacuum equation irrespective of the choice of coordinate  $y$ . Thus, if we use (71) is not necessary to assume that  $S(y)$  is given by (54). As we shall see in the sequel, it turns out that the metric given by (50), (51), (69), and (71) is quite special, since it admits a covariantly constant bivector whose square is the metric.

But, before proceeding, note that since  $A_1$  and  $A_2$  have opposite signs it follows that this metric cannot have Lorentzian signature. Indeed, by means of studying the reality conditions [32] of the null tetrad (8), one can see that: If  $\ell^2 < 0$  the signature is split (neutral), while if  $\ell^2 > 0$  we have that the signature is Euclidian for  $\Delta_1 \Delta_2 > 0$  and split for  $\Delta_1 \Delta_2 < 0$ . Furthermore, the bivectors  $\mathbf{l} \wedge \mathbf{n}$  and  $\mathbf{m}_1 \wedge \mathbf{m}_2$  are both real if  $\ell^2 < 0$  and both imaginary if  $\ell^2 > 0$ . Therefore, it is useful to separate our analysis in two cases.

Let us start considering the case  $\ell^2 > 0$ . In this case we have that the following real bivector is covariantly constant:

$$\mathbf{\Omega} = -i (\mathbf{l} \wedge \mathbf{n} + \epsilon \mathbf{m}_1 \wedge \mathbf{m}_2) . \quad (72)$$

Where  $\epsilon = \pm 1$ , depending on the function  $S$  and on the patch of the coordinate  $x$ . More precisely, we have

$$\epsilon = \text{Sign} \left[ x \frac{S'}{S} \right] = \pm 1 . \quad (73)$$

The bivector  $\mathbf{\Omega}$  is anti-self-dual if  $\epsilon = 1$ , namely its Hodge dual is equal to the negative of itself, whereas if  $\epsilon = -1$  it follows that  $\mathbf{\Omega}$  is self-dual, *i.e.*, its Hodge dual is equal to itself.<sup>4</sup> Since we have that  $\Omega^{ac} \Omega_{cb} = -\delta_b^a$ , we say that the tensor  $\mathbf{\Omega}$  is an almost complex structure. Note that the vectors  $\mathbf{l}$ ,  $\mathbf{m}_1$ ,  $\mathbf{n}$  and  $\mathbf{m}_2$  are eigenvectors of  $\mathbf{\Omega}$  with eigenvalues  $\pm i$ ,

$$\Omega^a_b l^b = i l^a , \quad \Omega^a_b n^b = -i n^a , \quad \Omega^a_b m_1^b = -i \epsilon m_1^a , \quad \Omega^a_b m_2^b = i \epsilon m_2^a . \quad (74)$$

Moreover, irrespective of the sign of  $\epsilon$ , the eigenspaces of  $\mathbf{\Omega}$  form integrable distributions. Indeed, as a consequence of (14), it follows that the isotropic planes generated by  $\{\mathbf{l}, \mathbf{m}_1\}$ ,  $\{\mathbf{n}, \mathbf{m}_2\}$ ,  $\{\mathbf{l}, \mathbf{m}_2\}$  and  $\{\mathbf{n}, \mathbf{m}_1\}$  are all tangent to integrable foliations. Because of this, we say that such almost complex structure is integrable [38]. Then, since  $\mathbf{\Omega}$  is a closed form,  $d\mathbf{\Omega} = 0$ , this 2-form is named a Kähler form. Thus, the solution found here is a Kähler metric. Particularly, if  $\Lambda = 0$  we end up with a Ricci-flat Kähler metric, also known as a Calabi-Yau manifold. In addition, this space is also endowed with the following real conformal Killing-Yano tensor

$$\mathbf{C} = i S(y) (\mathbf{l} \wedge \mathbf{n} - \epsilon \mathbf{m}_1 \wedge \mathbf{m}_2) , \quad (75)$$

which is a self-dual bivector if  $\epsilon > 1$  and anti-self-dual if  $\epsilon < 1$ .

On the other hand, if  $\ell^2 < 0$  we have that the real covariantly constant bivector is given by

$$\tilde{\mathbf{\Omega}} = \mathbf{l} \wedge \mathbf{n} - \epsilon \mathbf{m}_1 \wedge \mathbf{m}_2 , \quad (76)$$

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<sup>4</sup>It is worth recalling that, locally, the distinction between self-dual and anti-self-dual forms is just a matter of convention, since by multiplying the volume-form by  $-1$  these labels get interchanged.

with  $\epsilon$  given again by (73). In this case we have that  $\check{\Omega}^{ac}\check{\Omega}_{cb} = \delta_b^a$ , so that  $\check{\Omega}$  is called an almost paracomplex structure [39]. Because of the integrability of the eigen-planes of this paracomplex structure we say that it is integrable. Furthermore, since  $\check{\Omega}$  is a closed form this 2-form is named a para-Kähler form, so that the metric represents a para-Kähler manifold [39]. When  $\ell^2 < 0$  we also have that the following real bivector

$$\check{C} = S(y) (\boldsymbol{l} \wedge \boldsymbol{n} + \epsilon \boldsymbol{m}_1 \wedge \boldsymbol{m}_2) \quad (77)$$

is a conformal Killing-Yano tensor.

In addition to these geometrical objects, the space described here admits four null bivectors that are solutions of source-free Maxwell equations irrespective of the sign of the constant  $\ell^2$ :

$$\boldsymbol{B}_1^+ = \frac{1}{S' \sqrt{\Delta_1 \Delta_2}} \boldsymbol{m}_2 \wedge \boldsymbol{n} \quad , \quad \boldsymbol{B}_2^+ = \frac{1}{S' \sqrt{\Delta_1 \Delta_2}} \boldsymbol{l} \wedge \boldsymbol{m}_1 \quad , \quad (78)$$

$$\boldsymbol{B}_1^- = \frac{1}{S' \sqrt{\Delta_1 \Delta_2}} \boldsymbol{m}_1 \wedge \boldsymbol{n} \quad , \quad \boldsymbol{B}_2^- = \frac{1}{S' \sqrt{\Delta_1 \Delta_2}} \boldsymbol{l} \wedge \boldsymbol{m}_2 \quad . \quad (79)$$

The bivectors  $\boldsymbol{B}_1^+$  and  $\boldsymbol{B}_2^+$  are self-dual, while  $\boldsymbol{B}_1^-$  and  $\boldsymbol{B}_2^-$  are anti-self-dual. Since these bivectors are closed and co-closed we say that they obey the source-free Maxwell equations. Actually, since the energy-momentum tensor associated to these Maxwell fields is zero, we can say that they provide solutions to Einstein-Maxwell equations.

In order to find possible singularities of the space it is useful to take a look at some curvature invariant scalars, *i.e.*, scalars that are constructed from full contractions of the curvature and its derivatives. Note, for instance, that the Weyl scalars are not curvature invariants, since they depend on the choice of the null tetrad basis. However, the following scalars are true curvature invariants:

$$R^{abcd} R_{abcd} = \frac{16}{3} \Lambda^2 + 24 \left( \frac{\tilde{b}}{S^3} \right)^2 \quad , \quad (80)$$

$$R^{abcd} R_{cdef} R^e{}_{ab} = \frac{80}{9} \Lambda^3 + 48 \left( \frac{\tilde{b}}{S^3} \right)^2 - 48 \left( \frac{\tilde{b}}{S^3} \right)^3 \quad , \quad (81)$$

where  $R_{abcd}$  stands for the Riemann tensor. Note that these scalars diverge in the points in which the function  $S(y)$  vanishes, hinting the existence of singularities in these points. Nevertheless, it is interesting noting that these divergences cease to exist if the constant  $\tilde{b}$  vanishes. With the aim of understanding the meaning of the condition  $\tilde{b} = 0$  let us compute the Weyl scalars of this space.

Since the space considered in the present section is type  $(D, D)$  according to the generalized Petrov classification, with  $\boldsymbol{l} \wedge \boldsymbol{m}_1$ ,  $\boldsymbol{n} \wedge \boldsymbol{m}_2$ ,  $\boldsymbol{l} \wedge \boldsymbol{m}_2$  and  $\boldsymbol{n} \wedge \boldsymbol{m}_1$  being repeated principal null bivectors [38], it follows that the only Weyl scalars that can be different from zero are  $\Psi_2^+$  and  $\Psi_2^-$ . One can check that their values depend on the sign of the parameter  $\ell^2$ . Indeed, if  $\ell^2 > 0$  we find that

$$\Psi_2^+ = -(1 + \epsilon) \frac{\Lambda}{6} + (1 - \epsilon) \frac{\tilde{b}}{2S^3} \quad \text{and} \quad \Psi_2^- = -(1 - \epsilon) \frac{\Lambda}{6} + (1 + \epsilon) \frac{\tilde{b}}{2S^3} \quad . \quad (82)$$

So, if  $\tilde{b} = 0$  and  $\epsilon = 1$  the space is self-dual, meaning that only the self-dual part of the Weyl tensor is different from zero, whereas if  $\tilde{b} = 0$  and  $\epsilon = -1$  the space is anti-self-dual. Analogously, if  $\Lambda = 0$  the space is anti-self-dual for  $\epsilon = 1$  and self-dual for  $\epsilon = -1$ . On the other hand, if  $\ell^2 < 0$  the values of  $\Psi_2^+$  and  $\Psi_2^-$  are interchanged. More explicitly, if  $\ell^2$  is negative we have that:

$$\Psi_2^+ = -(1 - \epsilon) \frac{\Lambda}{6} + (1 + \epsilon) \frac{\tilde{b}}{2S^3} \quad \text{and} \quad \Psi_2^- = -(1 + \epsilon) \frac{\Lambda}{6} + (1 - \epsilon) \frac{\tilde{b}}{2S^3} \quad . \quad (83)$$

Thus, when  $\ell^2 < 0$  and  $\epsilon = 1$  the space is self-dual if  $\Lambda = 0$  and anti-self-dual if  $\tilde{b} = 0$ . Analogously, if  $\ell^2 < 0$  and  $\epsilon = -1$  the space is anti-self-dual if  $\Lambda = 0$  and self-dual if  $\tilde{b} = 0$ . So, we conclude the condition  $\tilde{b} = 0$  that avoids the divergence of the curvature invariants (80) and (81) means geometrically that the Weyl tensor is either self-dual or anti-self-dual.

## 6 Integrating Einstein's Equation when $P'_1 = 0$ and $P'_2 = 0$

The aim of the present section is to integrate Einstein's vacuum equation for the metric (18) in the special case in which the functions  $P_1(x)$  and  $P_2(y)$  are both constant. In what follows we shall denote these constants by  $p_1$  and  $p_2$  respectively. In this case, we can redefine the coordinates  $x$  and  $y$  along with the functions  $\Delta_1$ ,  $\Delta_2$ ,  $A_1$  and  $A_2$  in such a way to make  $\Delta_1(x) = 1$  and  $\Delta_2(y) = 1$ . Adopting these redefined coordinates we end up with the following line element:

$$ds^2 = S \left[ \frac{-A_2}{(p_1 + p_2)^2} (d\tau + p_1 d\sigma)^2 + \frac{A_1}{(p_1 + p_2)^2} (d\tau - p_2 d\sigma)^2 + dx^2 + dy^2 \right] \quad (84)$$

where  $S = S_1(x) + S_2(y)$ . As anticipated in Sec. 2, this metric is type  $D$  regardless of any restriction on the functions  $A_1$  and  $A_2$ . Now, computing the Ricci tensor we find that  $R^x_y$  is given by the expression (52) with  $\Delta_1(x) = 1$ . Therefore, in order for Einstein's vacuum equation with a cosmological constant to be satisfied, either  $S_1(x)$  or  $S_2(y)$  might be constant. One could also have that both functions are constant, but let us postpone the analysis of this case. So, let us assume that  $S_1(x)$  is a constant denoted by  $s_1$  and that  $S_2(y)$  is a non-constant function of  $y$ .<sup>5</sup> For future convenience, let us define the function  $H(y)$ :

$$H(y) = S^{1/4} = [s_1 + S_2(y)]^{1/4}. \quad (85)$$

Then, imposing  $R_{ab} l^a l^b$  to vanish and assuming  $H(y)$  to be non-constant, we find that  $A_2(y)$  must have the following general form:

$$A_2(y) = a_2 \frac{(H')^2}{H^6}, \quad (86)$$

with  $a_2$  being an arbitrary non-zero constant. Assuming (86), we have that the eight components of Einstein's vacuum equation displayed in (34) are satisfied. It remains to impose  $R_{ab} m_1^a m_2^b = \Lambda$  and  $R_{ab} l^a n^b = -\Lambda$ . The first of these conditions imply that  $A_1$  is given by

$$A_1(x) = a_1 \cos^2(2bx + c), \quad (87)$$

where  $a_1$ ,  $b$  and  $c$  are constants. Inserting this expression for  $A_1$  into  $R_{ab} m_1^a m_2^b = \Lambda$  yields the following differential equation for  $H$ :

$$H'' = H \left( b^2 - \frac{\Lambda}{4} H^4 \right). \quad (88)$$

One can also prove that if (88) holds then the remaining equations  $R_{ab} m_1^a m_2^b = \Lambda$  and  $R_{ab} l^a n^b = -\Lambda$  are both obeyed. Particularly, if  $\Lambda = 0$  the general solution of (88) is given by

$$H(y) = a_3 e^{by} + a_4 e^{-by} \quad (\Lambda = 0), \quad (89)$$

where  $a_3$  and  $a_4$  are arbitrary constants. It is worth noting that if either  $a_3$  or  $a_4$  vanish then  $\Psi_2 = 0$  and the space is flat. For  $\Lambda \neq 0$ , any non-constant solution for the non-linear differential equation

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<sup>5</sup>The opposite case, in which  $S_2$  is constant and  $S_1$  is non-constant can be obtained from the case  $S'_1 = 0$  and  $S'_2 \neq 0$  by means of interchanging the coordinates  $x$  and  $y$ .

(88) will generate a metric that is solution of Einstein's vacuum equation. This solution turns out to admit the following Killing-Yano tensor:

$$\mathbf{Y} = i H^2 \mathbf{m}_1 \wedge \mathbf{m}_2 . \quad (90)$$

Note that we can easily get rid of some constants in the solution (84) by means of redefining the coordinates as follows:

$$\tilde{\tau} = \frac{\sqrt{a_2} (\tau + p_1 \sigma)}{p_1 + p_2} , \quad \tilde{\sigma} = 2 \frac{\sqrt{a_1} (\tau - p_2 \sigma)}{p_1 + p_2} , \quad \tilde{x} = 2x + \frac{c}{b} + \frac{\pi}{2b} , \quad \tilde{y} = \frac{1}{2} [H(y)]^2 . \quad (91)$$

With these coordinates, the solution just obtained is given by

$$ds^2 = - \left( \frac{H'}{H} \right)^2 d\tilde{\tau}^2 + \left( \frac{H'}{H} \right)^{-2} d\tilde{y}^2 + \tilde{y}^2 (d\tilde{x}^2 + \sin^2(b\tilde{x}) d\tilde{\sigma}^2) . \quad (92)$$

With  $H(y)$  being a non-constant solution of (88). Although (88) is a non-linear differential equation, we can transform this equation into a linear equation by means of using the coordinate  $\tilde{y}$ . Indeed, defining  $F(\tilde{y}) \equiv \left( \frac{H'}{H} \right)^2$  we find that (88) is equivalent to the differential equation

$$\tilde{y} \frac{dF}{d\tilde{y}} + F = b^2 - \Lambda \tilde{y}^2 , \quad (93)$$

whose general solution is

$$\left( \frac{H'}{H} \right)^2 = F(\tilde{y}) = b^2 - \frac{2m}{\tilde{y}} - \frac{\Lambda}{3} \tilde{y}^2 . \quad (94)$$

Where  $m$  is an integration constant. Therefore, the solution given by (92) along with (88) is just the Schwarzschild-(A)dS spacetime with a possible conical singularity. In terms of these coordinates the null tetrad (8) is given by:

$$l = - \frac{1}{\sqrt{2F}} (F d\tilde{\tau} - d\tilde{y}) , \quad (95)$$

$$n = - \frac{1}{\sqrt{2F}} (F d\tilde{\tau} + d\tilde{y}) , \quad (96)$$

$$\mathbf{m}_1 = - \frac{\tilde{y}}{\sqrt{2}} (\sin(b\tilde{x}) d\tilde{\sigma} - i d\tilde{x}) , \quad (97)$$

$$\mathbf{m}_2 = - \frac{\tilde{y}}{\sqrt{2}} (\sin(b\tilde{x}) d\tilde{\sigma} + i d\tilde{x}) . \quad (98)$$

In particular, by means of (90) and (97), we arrive at the following expression for the Killing-Yano tensor in these new coordinates:

$$\mathbf{Y} = 2 \tilde{y}^3 \sin(b\tilde{x}) d\tilde{x} \wedge d\tilde{\sigma} . \quad (99)$$

### 6.1 The case $S_1$ and $S_2$ constant

In order to obtain (86) it was assumed that  $S_2(y)$  is non-constant. Now, it is time to consider the case when the functions  $P_1$ ,  $P_2$ ,  $S_1$  and  $S_2$  are all constant, in which case the component  $R_{ab} l^a l^b$  is automatically zero and there is no constraint over  $A_2(y)$  at this stage. Actually, one can check that the eight components (34) of Einstein's vacuum equation are already satisfied. Then, the remaining



equations  $R_{ab} m_1^a m_2^b = \Lambda$  and  $R_{ab} l^a n^b = -\Lambda$  provide non-linear differential equations for  $A_1$  and  $A_2$  respectively whose general solutions are:

$$A_1(x) = a_1 \cos^2(x \sqrt{s\Lambda} + b_1) \quad \text{and} \quad A_2(y) = a_2 \cos^2(y \sqrt{s\Lambda} + b_2), \quad (100)$$

where the  $a$ 's, the  $b$ 's and  $s \equiv S$  are constants. The above solution is valid only for  $\Lambda \neq 0$ . Instead, if  $\Lambda = 0$  the equations  $R_{ab} m_1^a m_2^b = \Lambda$  and  $R_{ab} l^a n^b = -\Lambda$  imply that  $A_1(x)$  and  $A_2(y)$  are quadratic polynomials of  $x$  and  $y$  respectively, but in this case it turns out that the spacetime is flat. Indeed, this can be grasped from the fact that the only non-vanishing Weyl scalars in the general case are

$$\Psi_2^+ = -\frac{\Lambda}{3} \quad \text{and} \quad \Psi_2^- = -\frac{\Lambda}{3}, \quad (101)$$

so that if  $\Lambda = 0$  then the Ricci tensor and the Weyl tensor are both identically zero, which implies that the space is flat. Hence, let us just consider the case of non-zero cosmological constant. The results of this paragraph lead to the conclusion that the metric (84) with  $S$  being a non-zero constant and the functions  $A_1$  and  $A_2$  given by (100) is a solution of Einstein's vacuum equation with cosmological constant  $\Lambda$ . Such solution turns out to admit the following two Killing-Yano tensors:

$$\mathbf{Y}_1 = \mathbf{l} \wedge \mathbf{n} \quad , \quad \mathbf{Y}_2 = -i \mathbf{m}_1 \wedge \mathbf{m}_2. \quad (102)$$

A convenient choice of coordinates for the solution considered in the present subsection is:

$$\hat{\tau} = \frac{\sqrt{s\Lambda} a_2}{p_1 + p_2} (\tau + p_1 \sigma) \quad , \quad \hat{\sigma} = \frac{\sqrt{s\Lambda} a_1}{p_1 + p_2} (\tau - p_2 \sigma) \quad (103)$$

$$\hat{x} = x \sqrt{s\Lambda} + b_1 - \frac{\pi}{2} \quad , \quad \hat{y} = y \sqrt{s\Lambda} + b_2 - \frac{\pi}{2}. \quad (104)$$

With these coordinates, we conclude that the general solution of Einstein's vacuum equation for the metric (84) with both functions  $S_1$  and  $S_2$  being constant is given by:

$$ds^2 = \frac{1}{\Lambda} \left[ -\sin^2(\hat{y}) d\hat{\tau}^2 + d\hat{y}^2 + d\hat{x}^2 + \sin^2(\hat{x}) d\hat{\sigma}^2 \right]. \quad (105)$$

The latter space is just the product of the 2-dimensional (Anti-)de Sitter space with a sphere of radius  $\Lambda^{-1/2}$ ,  $(A)dS_2 \times S^2$ . This space can be seen as a double Wick rotated version of the Nariai spacetime [40]. In terms of these new coordinates, the KY tensors of Eq. (102) are given by

$$\mathbf{Y}_1 = \frac{\sin(\hat{y})}{\Lambda} d\hat{y} \wedge d\hat{\tau} \quad , \quad \mathbf{Y}_2 = \frac{\sin(\hat{x})}{\Lambda} d\hat{x} \wedge d\hat{\sigma}. \quad (106)$$

Since the spaces  $(A)dS_2$  and  $S^2$  are maximally symmetric spaces of dimension two it follows that they both admit three independent Killing vectors. Therefore, the metric (105) should have six Killing vector fields. Indeed, one can check that the following six 1-forms are independent Killing fields:

$$\mathbf{k}_1 = \sin^2(\hat{x}) d\hat{\sigma}, \quad (107)$$

$$\mathbf{k}_2 = \sin(\hat{\sigma}) d\hat{x} + \sin(\hat{x}) \cos(\hat{x}) \cos(\hat{\sigma}) d\hat{\sigma}, \quad (108)$$

$$\mathbf{k}_3 = \cos(\hat{\sigma}) d\hat{x} - \sin(\hat{x}) \cos(\hat{x}) \sin(\hat{\sigma}) d\hat{\sigma}, \quad (109)$$

$$\mathbf{k}_4 = \sin^2(\hat{y}) d\hat{\tau}, \quad (110)$$

$$\mathbf{k}_5 = \sinh(\hat{\tau}) d\hat{y} + \sin(\hat{y}) \cos(\hat{y}) \cosh(\hat{\tau}) d\hat{\tau}, \quad (111)$$

$$\mathbf{k}_6 = \cosh(\hat{\tau}) d\hat{y} + \sin(\hat{y}) \cos(\hat{y}) \sinh(\hat{\tau}) d\hat{\tau}. \quad (112)$$

It turns out that the Killing tensors generated by the square of the Killing-Yano tensors  $\mathbf{Y}_1$  and  $\mathbf{Y}_2$  are reducible, namely they can be written as linear combination of symmetrized products of the Killing vectors. Indeed, defining  $Q_{1ab} = Y_{1a}{}^c Y_{1cb}$  and  $Q_{2ab} = Y_{2a}{}^c Y_{2cb}$ , it is easy to check that

$$\mathbf{Q}_1 = \frac{1}{2\Lambda} (\mathbf{k}_6 \odot \mathbf{k}_6 - \mathbf{k}_5 \odot \mathbf{k}_5 - \mathbf{k}_4 \odot \mathbf{k}_4) \quad \text{and} \quad \mathbf{Q}_2 = -\frac{1}{2\Lambda} (\mathbf{k}_1 \odot \mathbf{k}_1 + \mathbf{k}_2 \odot \mathbf{k}_2 + \mathbf{k}_3 \odot \mathbf{k}_3). \quad (113)$$

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